OBJETIVO

ITA
Matemática
Livro do Professor

12

MÓDULO 45

TRIGONOMETRIA II

1. Considere a equação

$(3 - 2\cos^2 x)\left(1 + \text{tg}^2\frac{x}{2}\right) - 6\,\text{tg}\,\frac{x}{2} = 0.$

a) Determine todas as soluções x no intervalo $[0, \pi[$.

b) Para as soluções encontradas em a), determine cotg x.

RESOLUÇÃO:

a) $$(3 - 2\,.\cos^2 x)\,.\left(1 + \text{tg}^2\frac{x}{2}\right) - 6\,.\,\text{tg}\,\frac{x}{2} = 0 \Leftrightarrow$$

$$\Leftrightarrow (3 - 2\,.\cos^2 x)\,.\sec^2\frac{x}{2} - 6\,.\,\frac{\text{sen}\,\frac{x}{2}}{\cos\frac{x}{2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{3 - 2\,.\cos^2 x}{\cos^2\frac{x}{2}} - 6\,.\,\frac{\text{sen}\,\frac{x}{2}}{\cos\frac{x}{2}} = 0$$

$$\Leftrightarrow \frac{3 - 2\,.\cos^2 x - 6\,.\,\text{sen}\,\frac{x}{2}\,.\cos\frac{x}{2}}{\cos^2\frac{x}{2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{3 - 2\,.\cos^2 x - 3\,.\,\text{sen}\,x}{\cos^2\frac{x}{2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow 3 - 2\,.(1 - \text{sen}^2 x) - 3\,.\,\text{sen}\,x = 0, \text{com } \cos\frac{x}{2} \neq 0 \Leftrightarrow$$

$$\Leftrightarrow 2\,.\,\text{sen}^2 x - 3\,.\,\text{sen}\,x + 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow \text{sen}\,x = 1 \text{ ou sen}\,x = \frac{1}{2}$$

No intervalo [0; π [, resulta:

$$x = \frac{\pi}{6} \text{ ou } x = \frac{\pi}{2} \text{ ou } x = \frac{5\pi}{6}$$

b) **Sendo** $\text{cotg}\,x = \frac{\cos x}{\text{sen}\,x}$**, temos:**

1) **Para** $x = \frac{\pi}{6} \rightarrow$

$$\rightarrow \text{cotg}\,\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\text{sen}\,\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

2) **Para** $x = \frac{\pi}{2} \rightarrow \text{cotg}\,\frac{\pi}{2} = \frac{\cos\frac{\pi}{2}}{\text{sen}\,\frac{\pi}{2}} = \frac{0}{1} = 0$

3) **Para** $x = \frac{5\pi}{6} \rightarrow$

$$\rightarrow \text{cotg}\,\frac{5\pi}{6} = \frac{\cos\frac{5\pi}{6}}{\text{sen}\,\frac{5\pi}{6}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Respostas: a) $x = \frac{\pi}{6}$ ou $x = \frac{\pi}{2}$ ou $x = \frac{5\pi}{6}$

b) $\text{cotg}\,x = \sqrt{3}$ ou $\text{cotg}\,x = 0$ ou $\text{cotg}\,x = -\sqrt{3}$

2. Sobre a equação tg x + cotg x = 2 sen 6x podemos afirmar que:

a) Apresenta uma raiz no intervalo $0 < x < \frac{\pi}{4}$

b) Apresenta duas raízes no intervalo $0 < x < \frac{\pi}{2}$

c) Apresenta uma raíz no intervalo $\frac{\pi}{2} < x < \pi$

d) Apresenta uma raíz no intervalo $\pi < x < \frac{3\pi}{2}$

e) Não apresenta raízes reais.

RESOLUÇÃO:

Observamos que $\text{tg } x + \text{cotg } x = \text{tg } x + \frac{1}{\text{tg } x} \geq 2$ **ou**

$\text{tg } x + \text{cotg } x = \text{tg} + \frac{1}{\text{tg } x} \leq -2.$

Como $-2 \leq 2 \text{ sen } 6x \leq 2$ **a igualdade** $\text{tg } x + \text{cotg } x = 2 \text{ sen } 6x$ **somente é viavel se:**

(tg x = 1 e sen 6x = 1) ou (tg x = –1 e sen 6x = –1) mas

$(\text{tg } x = 1 \Leftrightarrow x = \frac{\pi}{4} + k\pi \Leftrightarrow$

$\Leftrightarrow \text{sen } 6x = \text{sen } 6\left(\frac{\pi}{4} + k\pi\right) = \text{sen}\left(\frac{3\pi}{2} + 6k\pi\right) \neq 1$

$\text{tg } x = -1 \Rightarrow x = \frac{3\pi}{4} + k\pi \Rightarrow$

$\Rightarrow \text{sen } 6x = \text{sen } 6\left(\frac{3\pi}{4} + k\pi\right) = \text{sen}\left(\frac{9\pi}{2} + 6k\pi\right) \neq -1$

Logo a equação tg x + cotg x = 2 . sen 6x não admite solução real.
Resposta: E

3. Seja a um número real não nulo, satisfazendo $-1 \leq a \leq 1$.

Se dois ângulos agudos de um triângulo são dados por arc sen a e arc sec $\frac{1}{a}$, então o seno trigonométrico do terceiro ângulo desse triângulo é igual a:

a) $\frac{1}{3}$ b) $\frac{1}{3}$ c) $\frac{\sqrt{3}}{2}$ d) 1 e) $\frac{\sqrt{2}}{2}$

RESOLUÇÃO:

Sejam α, β e γ os ângulos internos de um triângulo.

Se:

1. $\alpha = \text{arc sen} a \Leftrightarrow \text{sen } \alpha = a$
2. $\beta = \text{arc sec } \frac{1}{a} \Leftrightarrow \sec \beta = \frac{1}{a} \Leftrightarrow \cos \beta = a$ **então:**

$\text{sen } \alpha = \cos \beta \Leftrightarrow \alpha + \beta = \frac{\pi}{2}$ **(α, β agudos)**

Assim: $\gamma = \pi - (\alpha + \beta) = \frac{\pi}{2} \Leftrightarrow \text{sen } \gamma = 1$

Resposta: D

4. Num triângulo ABC considere conhecidos os ângulos $B\hat{A}C$ e $C\hat{B}A$ e a medida d do lado AB. Nestas condições, a área S deste triângulo é dada pela relação:

a) $S = \dfrac{d^2}{2\,\text{sen}(B\hat{A}C + C\hat{B}A)}$

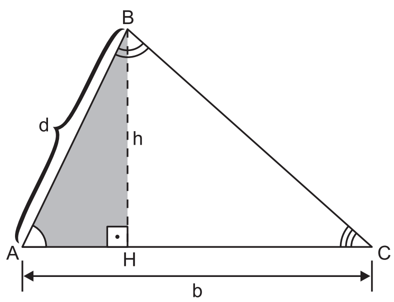
b) $S = \dfrac{d^2\,(\text{sen}\,B\hat{A}C)\,(\,\text{sen}\,C\hat{B}A)}{2\,\text{sen}(B\hat{A}C + C\hat{B}A)}$

c) $S = \dfrac{d^2\,\text{sen}\,C\hat{B}A}{2\,\text{sen}(B\hat{A}C + C\hat{B}A)}$

d) $S = \dfrac{d^2\,\text{sen}\,B\hat{A}C}{2\,\cos(B\hat{A}C + C\hat{B}A)}$

e) $S = \dfrac{d^2\,(\text{sen}\,B\hat{A}C)\,(\text{sen}\,C\hat{B}A)}{2\,\cos(B\hat{A}C + C\hat{B}A)}$

RESOLUÇÃO:



a) No ΔABH (retângulo em H), temos:
$\text{sen}\,B\hat{A}C\ \dfrac{h}{d} \Leftrightarrow h = d\,.\,(\text{sen}\,B\hat{A}C)$

b) Aplicando a Lei dos Senos no ΔABC, temos:

$$\frac{b}{\text{sen}\,C\hat{B}A} = \frac{d}{\text{sen}\,A\hat{C}B} \Leftrightarrow$$

$$\Leftrightarrow \frac{b}{\text{sen}\,C\hat{B}A} = \frac{d}{\text{sen}\,[180^o - (B\hat{A}C + C\hat{B}A)]} \Leftrightarrow$$

$$\Leftrightarrow \frac{b}{\text{sen}\,C\hat{B}A} = \frac{d}{\text{sen}\,(B\hat{A}C + C\hat{B}A)} \Leftrightarrow b = \frac{d\,.\,(\text{sen}\,C\hat{B}A)}{\text{sen}\,(B\hat{A}C + C\hat{B}A)}$$

$$\text{c) } S_{\Delta ABC} = \frac{b\,.\,h}{2} = \frac{\dfrac{d\,.\,(\text{sen}\,C\hat{B}A)}{\text{sen}\,(B\hat{A}C + C\hat{B}A)}\,.\,d\,(\text{sen}\,B\hat{A}C)}{2} \Leftrightarrow$$

$$\Leftrightarrow S_{\Delta ABC} = \frac{d_2\,.\,(\text{sen}\,B\hat{A}C)\,.\,(\text{sen}\,C\hat{B}A)}{2\,.\,\text{sen}\,(B\hat{A}C + C\hat{B}A)}$$

Resposta: B

MÓDULO 46

TRIGONOMETRIA II

1. **(ITA)** – O conjunto-solução de
$(tg^2x - 1)(1 - cotg^2x) = 4,\ x \neq k\pi/2,\ k \in \mathbb{Z}$, é:
a) $\{\pi/3 + k\pi/4,\ k \in \mathbb{Z}\}$
b) $\{\pi/4 + k\pi/4,\ k \in \mathbb{Z}\}$
c) $\{\pi/6 + k\pi/4,\ k \in \mathbb{Z}\}$
d) $\{\pi/8 + k\pi/4,\ k \in \mathbb{Z}\}$
e) $\{\pi/12 + k\pi/4,\ k \in \mathbb{Z}\}$

RESOLUÇÃO:

Para $x \neq k\,\dfrac{\pi}{2}$, $k \in \mathbb{Z}$, temos:

$$(tg^2x - 1)\,(1 - cotg^2x) = 4 \Leftrightarrow$$

$$\Leftrightarrow \frac{(\text{sen}^2x - \cos^2x)}{\cos^2x}\,.\,\frac{(\text{sen}^2x - \cos^2x)}{\text{sen}^2x} = 4 \Leftrightarrow$$

$$\Leftrightarrow (\text{sen}^2x - \cos^2x)^2 = 4\,\text{sen}^2x\,\cos^2x \Leftrightarrow$$

$$\Leftrightarrow \cos^2(2x) = \text{sen}^2(2x) \Leftrightarrow tg^2(2x) = 1 \Leftrightarrow$$

$$\Leftrightarrow tg\,(2x) = \pm 1 \Leftrightarrow 2x = \frac{\pi}{4} + k\,.\,\frac{\pi}{2},\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{\pi}{8} + k\,.\,\frac{\pi}{4},\ k \in \mathbb{Z}$$

O conjunto-solução da equação é:

$$\left\{\frac{\pi}{8} + k\,.\,\frac{\pi}{4},\ k \in \mathbb{Z}\right\}$$

Resposta: D

2. Prove que $\dfrac{tg^2 2\alpha - tg^2\alpha}{1 - tg^2 2\alpha \cdot tg^2\alpha} = tg^3\alpha \cdot tg\ \alpha$

RESOLUÇÃO:

$$\frac{tg^2 2\alpha - tg^2\alpha}{1 - tg^2 2\alpha \cdot tg^2\alpha} = \frac{(tg 2\alpha + tg\ \alpha).(tg 2\alpha - tg\ \alpha)}{(1 - tg 2\alpha \cdot tg\ \alpha).(1 + tg 2\alpha \cdot tg\ \alpha)} =$$

$$= \frac{(tg 2\alpha + tg\ \alpha)}{(1 - tg 2\alpha \cdot tg\ \alpha)} \cdot \frac{(tg 2\alpha - tg\ \alpha)}{(1 + tg 2\alpha \cdot tg\ \alpha)} = tg 3\alpha \cdot tg\ \alpha$$

Resposta: Demonstração

3. O valor do sen 47° + sen 61° – sen 11° – sen 25° é igual a:

a) cos 47° b) sen 25° c) cos 18°
d) sen 12° e) cos 7°

RESOLUÇÃO:

$\text{sen } 47^o + \text{sen } 61^o - \text{sen } 11^o - \text{sen } 25^o =$

$= (\text{sen } 61^o + \text{sen } 47^o) - (\text{sen } 25^o + \text{sen } 11^o) =$

$$= 2 \text{ sen } \frac{61^o + 47^o}{2} \cdot \cos \frac{61^o - 47^o}{2} - 2\text{sen } \frac{25^o + 11^o}{2} \cdot \cos \frac{25^o - 11^o}{2} =$$

$= 2\text{sen } 54^o \cdot \cos 7^o - 2\text{sen } 18 \cdot \cos 7^o =$

$= 2 \cdot \cos 7^o \cdot (\text{sen } 54^o - \text{sen } 18^o) =$

$$= 2 \cdot \cos 7^o \cdot 2 \text{ sen} \left(\frac{54^o - 18^o}{2}\right) \cdot \cos \left(\frac{54^o + 18^o}{2}\right) =$$

$= 2 \cdot \cos 7^o \cdot 2 \text{ sen } 18^o. \cos 36^o =$

$$= 2 \cdot \cos 7^o \frac{2 \cdot \text{sen } 18^o \cdot \cos 18^o \cdot \cos 36^o}{\cos 18^o} =$$

$$= 2 \cdot \cos 7^o \cdot \frac{\text{sen}36^o \cos36^o}{\cos18^o} = \cos 7^o \cdot \frac{2 \cdot \text{sen}36^o \cos36^o}{\cos18^o} =$$

$$= \cos 7^o \cdot \frac{\text{sen}72^o}{\cos18^o} = \cos 7 \cdot 1 = \cos 7^o, \text{ pois sen } 72^o = \cos 18^o$$

Resposta: E

MÓDULO 47

TRIGONOMETRIA II

1. A equação [sen (cos x)] . [cos (sen x)] = 1 é satisfeita para

a) $x = \dfrac{\pi}{4}$.

b) x = 0.

c) nenhum valor de x.

d) todos os valores de x.

e) todos os valores de x pertencentes ao terceiro quadrante.

RESOLUÇÃO:

$[\text{sen}(\cos x)] \cdot [\cos(\text{sen } x)] = 1 \Leftrightarrow (\text{sen}(\cos x) = 1 \text{ e } \cos(\text{sen } x) = 1) \text{ ou}$

$(\text{sen}(\cos x) = -1 \text{ e } \cos(\text{sen } x) = -1) \Leftrightarrow \left(\cos x = \dfrac{\pi}{2} + 2k\pi \text{ e sen } x = 2p\pi\right)$

$\text{ou } \left(\cos x = \dfrac{3\pi}{2} + 2m\pi \text{ e sen } x = \pi + 2n\pi\right), \text{ com } k, p, m, n \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow \nexists x$, pois, para qualquer valor de k, p, m, n $\in \mathbb{Z}$, tem-se $\cos x \notin [-1;1]$.

Resposta: C

2. Sabendo que $tg^2\left(x + \frac{1}{6}\pi\right) = \frac{1}{2}$, para algum $x \in \left[0, \frac{1}{2}\pi\right]$, determine sen x.

RESOLUÇÃO:

$$tg^2\left(x+\frac{\pi}{6}\right) = \frac{1}{2} \Rightarrow tg\left(x+\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}, \text{ pois:}$$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le \frac{2\pi}{3}$$

Assim:

$$\frac{tg\,x + tg\left(\frac{\pi}{6}\right)}{1 - tg\left(\frac{\pi}{6}\right)\,tg\,x} = \frac{\sqrt{2}}{2} \Leftrightarrow \frac{tg\,x + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}\,.\,tg\,x} = \frac{\sqrt{2}}{2} \Leftrightarrow$$

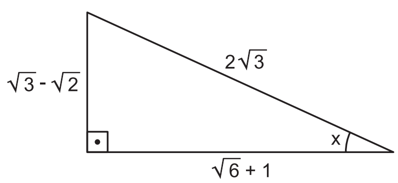
$$\Leftrightarrow 2\,tg\,x + \frac{2\sqrt{3}}{3} = \sqrt{2} - \frac{\sqrt{6}}{3}\,tg\,x \Leftrightarrow$$

$$\Leftrightarrow 6\,tg\,x + 2\sqrt{3} = 3\sqrt{2} - \sqrt{6}\,tg\,x \Leftrightarrow$$

$$\Leftrightarrow (6 + \sqrt{6})\,tg\,x = 3\sqrt{2} - 2\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow tg\,x = \frac{3\sqrt{2} - 2\sqrt{3}}{6 + \sqrt{6}} \Leftrightarrow tg\,x = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{6} + 1}$$

2) Como $0 < x < \frac{\pi}{2}$, podemos então montar o seguinte triângulo retângulo:



do qual podemos concluir que:

$$sen\,x = \frac{\sqrt{3} - \sqrt{2}}{2\sqrt{3}} \Leftrightarrow sen\,x = \frac{3 - \sqrt{6}}{6}$$

Resposta: $\frac{3 - \sqrt{6}}{6}$

3. O valor de
$tg^{10}x - 5tg^8x\,sec^2x + 10tg^6x\,sec^4x - 10tg^4x\,sec^6x +$
$+ 5tg^2x\,sec^8x - sec^{10}x$, para todo $x \in \left[0, \frac{\pi}{2}\right[$, é:

a) 1 b) $\frac{-\,sec^2x}{1 + sen^2x}$ c) – sec x + tg x
d) – 1 e) zero

RESOLUÇÃO:

Para $x \in \left[0; \frac{\pi}{2}\right[$ temos:

$$tg^{10}x - 5tg^8x\,sec^2x + 10tg^6x\,sec^4x - 10\,tg^4x\,sec^6x +$$
$$+ 5tg^2x\,sec^8x - sec^{10}x = (tg^2x - sec^2x)^5 =$$

$$= \left(\frac{sen^2x}{cos^2x} - \frac{1}{cos^2x}\right)^5 = \left(\frac{sen^2x - 1}{cos^2x}\right)^5 = \left(\frac{-\,cos2x}{cos^2x}\right)^5 =$$

$$= (-1)^5 = -1$$

Resposta: D

MÓDULO 48

TRIGONOMETRIA II

1. A equação em x,

$\operatorname{arctg}(e^x + 2) - \operatorname{arccotg}\left(\dfrac{e^x}{e^{2x} - 1}\right) = \dfrac{\pi}{4}, x \in \mathbb{R} \setminus \{0\}$,

a) admite infinitas soluções, todas positivas.
b) admite uma única solução, e esta é positiva.
c) admite três soluções que se encontram no intervalo $\left]-\dfrac{5}{2}, \dfrac{3}{2}\right[$.
d) admite apenas soluções negativas.
e) não admite solução.

RESOLUÇÃO:

Com $-\dfrac{\pi}{2} < a < \dfrac{\pi}{2}$ **e** $0 < b < \pi$, **temos:**

1) $a = \operatorname{arc\,tg}(e^x + 2) \Leftrightarrow \operatorname{tg} a = e^x + 2$

2) $b = \operatorname{arc\,cotg}\left(\dfrac{e^x}{e^{2x} - 1}\right) \Leftrightarrow \operatorname{cotg} b = \dfrac{e^x}{e^{2x} - 1} \Leftrightarrow$

$\Leftrightarrow \operatorname{tg} b = \dfrac{e^{2x} - 1}{e^x}$

3) $a - b = \dfrac{\pi}{4} \Leftrightarrow \operatorname{tg}(a - b) = \operatorname{tg}(\pi/4) \Leftrightarrow$

$\Leftrightarrow \dfrac{\operatorname{tg} a - \operatorname{tg} b}{1 + \operatorname{tg} a \,.\, \operatorname{tg} b} = 1 \Leftrightarrow \operatorname{tg} a - \operatorname{tg} b = 1 + \operatorname{tg} a \,.\, \operatorname{tg} b$

Se, na equação:

$(e^x + 2) - \left(\dfrac{e^{2x} - 1}{e^x}\right) = 1 + (e^x + 2) \,.\, \left(\dfrac{e^{2x} - 1}{e^x}\right)$

fizermos $e^x = y$**, resulta:**

$(y + 2) - \left(\dfrac{y^2 - 1}{y}\right) = 1 + (y + 2) \,.\, \left(\dfrac{y^2 - 1}{y}\right) \Leftrightarrow$

$\Leftrightarrow y^2 + 2y - y^2 + 1 = y + y^3 - y + 2y^2 - 2 \Leftrightarrow$

$\Leftrightarrow y^3 + 2y^2 - 2y - 3 = 0 \Leftrightarrow (y + 1) \,.\, (y^2 + y - 3) = 0 \Leftrightarrow$

$\Leftrightarrow y = -1$ **ou** $y = \dfrac{-1 + \sqrt{13}}{2}$ **ou** $y = \dfrac{-1 - \sqrt{13}}{2}$

Como $y > 0$**, a única possibilidade e**

$y = \dfrac{-1 + \sqrt{13}}{2}$

Portanto:

$e^x = \dfrac{\sqrt{13} - 1}{2} \Leftrightarrow x = \log_e\left(\dfrac{\sqrt{13} - 1}{2}\right) > 0$

Dessa forma, a equação admite uma única solução, e esta é positiva.

2. O valor da soma $\displaystyle\sum_{n=1}^{6} \operatorname{sen}\left(\frac{2\alpha}{3^n}\right) \operatorname{sen}\left(\frac{\alpha}{3^n}\right)$, para todo $\alpha \in \mathbb{R}$, é igual a

a) $\dfrac{1}{2}\left[\cos\left(\dfrac{\alpha}{729}\right) - \cos\alpha\right]$.

b) $\dfrac{1}{2}\left[\operatorname{sen}\left(\dfrac{\alpha}{243}\right) - \operatorname{sen}\left(\dfrac{\alpha}{729}\right)\right]$.

c) $\cos\left(\dfrac{\alpha}{243}\right) - \cos\left(\dfrac{\alpha}{729}\right)$.

d) $\dfrac{1}{2}\left[\cos\left(\dfrac{\alpha}{729}\right) - \cos\left(\dfrac{\alpha}{243}\right)\right]$.

e) $\cos\left(\dfrac{\alpha}{729}\right) - \cos\alpha$.

RESOLUÇÃO:

Lembrando que $\cos(a + b) - \cos(a - b) = -2 \operatorname{sen} a \,.\, \operatorname{sen} b$**, temos:**

$\cos\left(\dfrac{2\alpha}{3^n} + \dfrac{\alpha}{3^n}\right) - \cos\left(\dfrac{2\alpha}{3^n} - \dfrac{\alpha}{3^n}\right) =$

$= -2 \operatorname{sen}\left(\dfrac{2\alpha}{3^n}\right) \,.\, \operatorname{sen}\left(\dfrac{\alpha}{3^n}\right) \Leftrightarrow$

$\Leftrightarrow \cos\left(\dfrac{3\alpha}{3^n}\right) - \cos\left(\dfrac{\alpha}{3^n}\right) = -2\operatorname{sen}\left(\dfrac{2\alpha}{3^n}\right) \,.\, \operatorname{sen}\left(\dfrac{\alpha}{3^n}\right) \Leftrightarrow$

$\Leftrightarrow \operatorname{sen}\left(\dfrac{2\alpha}{3^n}\right) \,.\, \operatorname{sen}\left(\dfrac{\alpha}{3^n}\right) = \dfrac{1}{2}\left[\cos\left(\dfrac{\alpha}{3^n}\right) - \cos\left(\dfrac{3\alpha}{3^n}\right)\right] \Leftrightarrow$

Desta forma:

$$\sum_{n=1}^{6} \text{sen}\left(\frac{2\alpha}{3^n}\right).\text{sen}\left(\frac{\alpha}{3^n}\right) = \sum_{n=1}^{6} \frac{1}{2}\left[\cos\left(\frac{\alpha}{3^n}\right) - \cos\left(\frac{3\alpha}{3^n}\right)\right] =$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{3}\right) - \cos\alpha + \cos\left(\frac{\alpha}{9}\right) - \cos\left(\frac{\alpha}{3}\right) + \right.$$

$$+ \cos\left(\frac{\alpha}{27}\right) - \cos\left(\frac{\alpha}{9}\right) + \cos\left(\frac{\alpha}{81}\right) - \cos\left(\frac{\alpha}{27}\right) +$$

$$\left. + \cos\left(\frac{\alpha}{243}\right) - \cos\left(\frac{\alpha}{81}\right) + \cos\left(\frac{\alpha}{729}\right) - \cos\left(\frac{\alpha}{243}\right)\right] =$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{729}\right) - \cos\alpha\right]$$

3. **(IME)** – Resolva a equação $\text{tg}\,\alpha + \text{tg}\,(2\alpha) = 2\,\text{tg}\,(3\alpha)$, sabendo-se que $\alpha \in [0, \pi/2)$.

RESOLUÇÃO:

$\text{tg}\,\alpha + \text{tg}\,(2\alpha) = 2\text{tg}\,(3\alpha) \Rightarrow \text{tg}\,\alpha + \text{tg}\,(2\alpha) = 2\,\text{tg}\,(\alpha + 2\alpha) \Rightarrow$

$$\Rightarrow (\text{tg}\,\alpha + \text{tg}\,2\alpha) = 2\,.\,\frac{\text{tg}\,\alpha + \text{tg}(2\alpha)}{1 - \text{tg}\,\alpha\,.\,\text{tg}(2\alpha)} \Leftrightarrow$$

$\Leftrightarrow \text{tg}\,\alpha + \text{tg}\,2\alpha = 0$ ou $\text{tg}\,\alpha\,.\,\text{tg}(2\alpha) = -1$

1) $\text{tg}\,\alpha + \text{tg}\,2\alpha = 0 \Rightarrow \text{tg}\,\alpha + \dfrac{2\text{tg}\,\alpha}{1 - \text{tg}^2\alpha} = 0 \Rightarrow$

$$\Rightarrow \frac{-\text{tg}^3\alpha + 3\text{tg}\,\alpha}{1 - \text{tg}^2\alpha} = 0 \Leftrightarrow \text{tg}^3\alpha - 3\text{tg}\,\alpha = 0 \Leftrightarrow$$

$\Leftrightarrow \text{tg}\,\alpha = 0$ ou $\text{tg}^2\alpha = 3 \Leftrightarrow \text{tg}\,\alpha = 0$ ou $\text{tg}\,\alpha = \pm\sqrt{3}$

$\Leftrightarrow \alpha = 0$ ou $\alpha = \dfrac{\pi}{3}$, pois $\alpha \in \left[0; \dfrac{\pi}{2}\right)$

2) $\text{tg}\,\alpha\,.\,\text{tg}(2\alpha) = -1 \Leftrightarrow \text{tg}\,\alpha\,.\,\dfrac{2\text{tg}\,\alpha}{1 - \text{tg}^2\alpha} = -1 \Leftrightarrow$

$\Leftrightarrow 2\text{tg}^2\alpha = -1 + \text{tg}^2\alpha \Leftrightarrow \text{tg}^2\alpha = -1 \Rightarrow \nexists\alpha$, pois $\text{tg}^2\alpha \geq 0$

Resposta: $\left\{0; \dfrac{\pi}{3}\right\}$

4. Resolva a equação $2\,\text{sen}\,11x + \cos 3x + \sqrt{3}\,\text{sen}\,3x = 0$.

RESOLUÇÃO:

$2\text{sen}\,11x + \cos 3x + \sqrt{3}\,\text{sen}\,3x = 0 \Leftrightarrow$

$\Leftrightarrow 2\text{sen}\,11x = -\cos 3x - \sqrt{3}\,\text{sen}\,3x \Leftrightarrow$

$\Leftrightarrow \text{sen}\,11x = -\dfrac{1}{2}\cos 3x - \dfrac{\sqrt{3}}{2}\,\text{sen}\,3x \Leftrightarrow$

$\Leftrightarrow \text{sen}\,11x = -\left[\text{sen}\,\dfrac{\pi}{6}\,.\cos 3x + \cos\dfrac{\pi}{6}\,.\,\text{sen}\,3x\right] \Leftrightarrow$

$\Leftrightarrow \text{sen}\,11x = -\,\text{sen}\left(\dfrac{\pi}{6} + 3x\right) \Leftrightarrow \text{sen}\,11x = \text{sen}\left(-\dfrac{\pi}{6} - 3x\right) \Leftrightarrow$

$\Leftrightarrow 11x = -\dfrac{\pi}{6} - 3x + 2k\pi$ ou $11x = \pi - \left(-\dfrac{\pi}{6} - 3x\right) + 2k\pi \Leftrightarrow$

$\Leftrightarrow x = -\dfrac{\pi}{84} + \dfrac{k\pi}{7}$ ou $x = \dfrac{7\pi}{48} + \dfrac{k\pi}{4}$, com $k \in \mathbb{Z}$

Resposta:

$$V = \left\{x \in \mathbb{R} \,\middle|\, x = -\frac{\pi}{84} + \frac{k\pi}{7} \text{ ou } x = \frac{7\pi}{48} + \frac{k\pi}{4}, \text{com } k \in \mathbb{Z}\right\}$$

exercícios-tarefa

■ MÓDULO 45

1. **(ITA)** – Seja **a** um número real tal que $a \neq \frac{\pi}{2} + k \cdot \pi$, em que $k \in \mathbb{Z}$. Se $(x_0; y_0)$ é solução do sistema

$$\begin{cases} (2 \sec a) \cdot x + (3 \operatorname{tg} a) \cdot y = 2 \cdot \cos a \\ (2 \operatorname{tg} a) \cdot x + (3 \sec a) \cdot y = 0 \end{cases},$$

então podemos afirmar que

a) $x_0 + y_0 = 3 - 2 \operatorname{sen} a$

b) $\left(\frac{2}{3} x_0\right)^2 - (y_0)^2 = \frac{4}{9} \cdot \cos^2 a + 2$

c) $x_0 - y_0 = 0$

d) $x_0 + y_0 = 0$

e) $\left(\frac{2}{3} x_0\right)^2 - (y_0)^2 = \frac{4}{9} \cdot \cos^2 a$

2. A expressão $\frac{\operatorname{sen} \theta}{1 + \cos \theta}$, 0 < θ , π, é idêntica a

a) $\sec \frac{\theta}{2}$ b) $\operatorname{cosec} \frac{\theta}{2}$ c) $\operatorname{cotg} \frac{\theta}{2}$

d) $\operatorname{tg} \frac{\theta}{2}$ e) $\frac{\theta}{2}$

■ MÓDULO 46

1. Mostre que $\operatorname{sen} 18^\circ \cdot \cos 36^\circ = \frac{1}{4}$.

■ MÓDULO 47

1. O número de raízes reais da equação

$\sum_{n=1}^{5} (\cos x)^{2n} = 5$, no intervalo $[0; 4\pi]$, é

a) 2 b) 3 c) 4 d) 5 e) 6

2. Se os números reais α e β, com $\alpha + \beta = \frac{4\pi}{3}$, $0 \leq \alpha \leq \beta$, maximizam a soma sen α + sen β, então α é igual a

a) $\frac{\pi\sqrt{3}}{3}$. b) $\frac{2\pi}{3}$. c) $\frac{3\pi}{5}$. d) $\frac{5\pi}{8}$. e) $\frac{7\pi}{12}$.

■ MÓDULO 48

1. Resolver em ℝ, a equação

$5\operatorname{sen}^2 x + \sqrt{3}\operatorname{sen} x \cdot \cos x + 6\cos^2 x = 5$

2. Resolver, em ℝ, a equação $\arccos x - \operatorname{arcsen} x = \frac{\pi}{6}$

resolução dos exercícios-tarefa

■ MÓDULO 45

1) $\begin{cases} (2 \sec a)x + (3 \operatorname{tg} a)y = 2 \cos a \\ (2 \operatorname{tg} a)x + (3 \sec a)\, y = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} 4 \sec^2 a\, x^2 + 12 \sec a \cdot \operatorname{tg} a\, xy + 9 \operatorname{tg}^2 a\, y^2 = 4 \cos^2 a \\ 4 \operatorname{tg}^2 a\, x^2 + 12 \sec a \cdot \operatorname{tg} a\, xy + 9 \sec^2 a\, y^2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow 4(\sec^2 a - \operatorname{tg}^2 a)x^2 + 9(\operatorname{tg}^2 a - \sec^2 a)y^2 = 4 \cos^2 a \Rightarrow$

$\Rightarrow 4x^2 - 9y^2 = 4 \cos^2 a \Leftrightarrow \left(\frac{2}{3}x\right)^2 - y^2 = \frac{4}{9}\cos^2 a$

Se $(x_0;y_0)$ é solução do sistema, então

$\left(\frac{2}{3}x_0\right)^2 - (y_0)^2 = \frac{4}{9}\cos^2 a$

Resposta: E

2) Sabe-se que:

$\operatorname{sen} \theta = 2 \cdot \operatorname{sen} \frac{\theta}{2} \cdot \cos \frac{\theta}{2}$

$\cos \theta = \cos^2 \frac{\theta}{2} - \operatorname{sen}^2 \frac{\theta}{2}$

$\operatorname{sen}^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = 1$

Assim, para $0 < \theta < \pi \Leftrightarrow 0 < \frac{\theta}{2} < \frac{\pi}{2}$**, tem-se:**

$$\frac{\operatorname{sen} \theta}{1 + \cos \theta} = \frac{2 \cdot \operatorname{sen} \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{\operatorname{sen}^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} - \operatorname{sen}^2 \frac{\theta}{2}} =$$

$$= \frac{2 \cdot \operatorname{sen} \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{2 \cdot \cos^2 \frac{\theta}{2}} = \frac{\operatorname{sen} \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \operatorname{tg} \frac{\theta}{2}$$

Resposta: D

■ MÓDULO 46

1) $\text{sen } 18° \,.\, \cos 36° = \dfrac{\text{sen } 18° \,.\, \cos 18° \,.\, \cos 36°}{\cos 18°} =$

$= \dfrac{1}{2} \,.\, \dfrac{2 \,.\, \text{sen } 18° \,.\, \cos 18° \,.\, \cos 36°}{\cos 18°} =$

$= \dfrac{1}{2} \,.\, \dfrac{1}{2} \,.\, \dfrac{2 \,.\, \text{sen } 36° \cos 36°}{\cos 18°} = \dfrac{1}{4} \,.\, \dfrac{\text{sen } 72°}{\cos 18°} =$

$= \dfrac{1}{4} \,.\, 1 = \dfrac{1}{4}$, pois $\text{sen } 72° = \cos 18°$

Resposta: Demonstração

■ MÓDULO 47

1) Como $0 \le (\cos x)^{2n} \le 1$, tem-se que

$\sum\limits_{n=1}^{5} (\cos x)^{2n} = 5 \Leftrightarrow (\cos x)^2 = 1 \Leftrightarrow \cos x = \pm 1 \Leftrightarrow x = 0$,

$x = \pi$, $x = 2\pi$, $x = 3\pi$ ou $x = 4\pi$, pois $x \in [0; 4\pi]$

Resposta: D

2)

1) $\left.\begin{array}{r} \text{sen } \alpha + \text{sen } \beta = 2 \text{ sen } \dfrac{\alpha+\beta}{2} \cos \dfrac{\alpha-\beta}{2} \\ \alpha + \beta = \dfrac{4\pi}{3} \end{array}\right\} \Rightarrow$

$\Rightarrow \text{sen } \alpha + \text{sen } \beta = \sqrt{3} \cos \dfrac{\alpha - \beta}{2} = \sqrt{3} \cos \left(\alpha - \dfrac{2\pi}{3}\right)$

2) $\text{sen } \alpha + \text{sen } \beta = \sqrt{3} \,.\, \cos \left(\alpha - \dfrac{2\pi}{3}\right)$ é máximo para

$\alpha - \dfrac{2\pi}{3} = 0 \Rightarrow \alpha = \dfrac{2\pi}{3}$

■ MÓDULO 48

1) $5\text{sen}^2 x + \sqrt{3}\text{sen } x \,.\, \cos x + 6\cos^2 x = 5 \Leftrightarrow$

$\Leftrightarrow 5.(1 - \cos^2 x) + \sqrt{3}\text{sen } x \,.\, \cos x + 6\cos^2 x = 5 \Leftrightarrow$

$\Leftrightarrow \sqrt{3}\text{sen } x \,.\, \cos x + \cos^2 x = 0 \Leftrightarrow$

$\Leftrightarrow \cos x = 0$ ou $\sqrt{3}\text{sen } x + \cos x = 0 \Leftrightarrow$

$\Leftrightarrow \cos x = 0$ ou $\text{tg } x = -\dfrac{\sqrt{3}}{3} \Leftrightarrow$

$\Leftrightarrow x = \dfrac{\pi}{2} + k\pi$ ou $x = -\dfrac{\pi}{3} + k\pi$

Resposta:

$V = \{x \in \mathbb{R} \mid x = \dfrac{\pi}{2} + k\pi$ ou $x = -\dfrac{\pi}{3} + k\pi$, com $k \in \mathbb{N}\}$

2) Fazendo $a = \arccos x$ temos $\cos a = x$, com $0 \le a \le \pi$ e $\text{sen } a = \sqrt{1 - x^2}$.

Fazendo $b = \text{arcsen } x$ temos $\text{sen } b = x$, com $-\dfrac{\pi}{2} \le b \le \dfrac{\pi}{2}$ e $\cos b = \sqrt{1 - x^2}$.

Desta forma, $\arccos x - \text{arcsen } x = \dfrac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow a - b = \dfrac{\pi}{6} \Leftrightarrow \cos(a - b) = \cos \dfrac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow \cos a \,.\, \cos b + \text{sen } a \,.\, \text{sen } b = \dfrac{\sqrt{3}}{2} \Leftrightarrow$

$\Leftrightarrow x \,.\, \sqrt{1 - x^2} + \sqrt{1 - x^2} \,.\, x = \dfrac{\sqrt{3}}{2} \Leftrightarrow$

$\Leftrightarrow 4.x.\sqrt{1 - x^2} = \sqrt{3} \Leftrightarrow 16x^2(1 - x^2) = 3 \Leftrightarrow$

$\Leftrightarrow 16x^4 - 16x^2 + 3 = 0 \Leftrightarrow x = \dfrac{1}{2}$ ou $x = -\dfrac{1}{2}$ ou

$x = \dfrac{\sqrt{3}}{2}$ ou $x = -\dfrac{\sqrt{3}}{2}$

Como durante a resolução tivemos que elevar a equação ao quadrado, devemos experimentar as respostas obtidas.

Para $x = \dfrac{1}{2} \Leftrightarrow \arccos x - \text{arcsen } x =$

$= \arccos\left(\dfrac{1}{2}\right) - \text{arcsen}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3} - \dfrac{\pi}{6} = \dfrac{\pi}{6}$

Para $x = -\dfrac{1}{2} \Leftrightarrow \arccos x - \text{arcsen } x =$

$= \arccos\left(-\dfrac{1}{2}\right) - \text{arcsen}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3} - \left(-\dfrac{\pi}{6}\right) =$

$= \dfrac{5\pi}{6} \ne \dfrac{\pi}{6}$

Para $x = \dfrac{\sqrt{3}}{2} \Leftrightarrow \arccos x - \text{arcsen } x =$

$= \arccos\left(\dfrac{\sqrt{3}}{2}\right) - \text{arcsen}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6} - \left(\dfrac{\pi}{3}\right) =$

$= -\dfrac{\pi}{6} \ne \dfrac{\pi}{6}$

Para $x = -\dfrac{\sqrt{3}}{2} \Leftrightarrow \arccos x - \text{arcsen } x =$

$= \arccos\left(-\dfrac{\sqrt{3}}{2}\right) - \text{arcsen}\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6} - \left(-\dfrac{\pi}{3}\right) =$

$= \dfrac{7\pi}{6} \ne \dfrac{\pi}{6}$, portanto, apenas $x = \dfrac{1}{2}$ é solução.

Respostas: $V = \left\{\dfrac{1}{2}\right\}$