FÓRMULAS DE ADIÇÃO, SUBTRAÇÃO, ARCO DOBRO, TRIPLO, METADE E PROSTAFÉRESE

1. FÓRMULAS DE ARCO SOMA E DIFERENÇA

As fórmulas a seguir permitem calcular o seno e o cosseno da soma e da diferença de arcos.

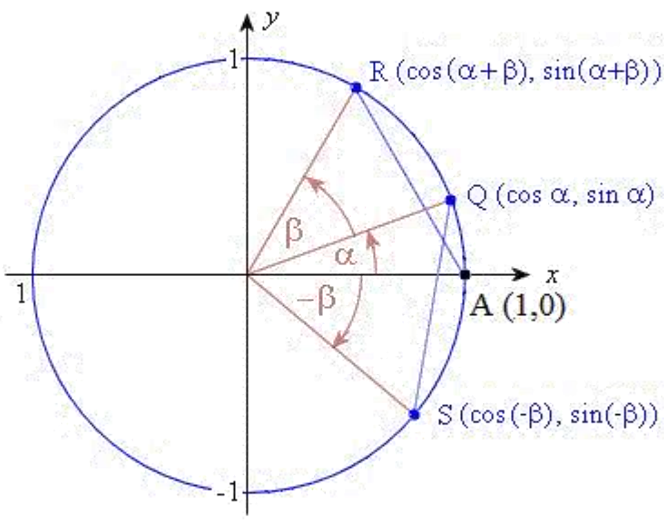
$$\text{sen}(\alpha+\beta)=\text{sen}\alpha\cdot\cos\beta+\text{sen}\beta\cdot\cos\alpha$$

$$\text{sen}(\alpha-\beta)=\text{sen}\alpha\cdot\cos\beta-\text{sen}\beta\cdot\cos\alpha$$

$$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\text{sen}\alpha\cdot\text{sen}\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\text{sen}\alpha\cdot\text{sen}\beta$$

Demonstração:



Sejam Q, R e S a imagem no ciclo trigonométrico de arcos com primeira determinação positiva α, $(\alpha+\beta)$ e $(-\beta)$, respectivamente. Logo, $AR=QS$, o que implica $\overline{AR}=\overline{QS}$.

As coordenadas desses pontos são dadas por:

$Q=(\cos\alpha,\text{sen}\alpha)$, $R(\cos(\alpha+\beta),\text{sen}(\alpha+\beta))$ e $S=(\cos(-\beta),\text{sen}(-\beta))=(\cos\beta,-\text{sen}\beta)$.

Aplicando a fórmula da distância entre pontos, temos:

$$\overline{AR}=\overline{QS}\Leftrightarrow\sqrt{(\cos(\alpha+\beta)-1)^2+(\text{sen}(\alpha+\beta)-0)^2}=\sqrt{(\cos\alpha-\cos\beta)^2+(\text{sen}\alpha-(-\text{sen}\beta))^2}$$

$$\Leftrightarrow\cos^2(\alpha+\beta)-2\cos(\alpha+\beta)+1+\text{sen}^2(\alpha+\beta)=\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\text{sen}^2\alpha+2\text{sen}\alpha\text{sen}\beta+\text{sen}^2\beta$$

$$\Leftrightarrow 2-2\cos(\alpha+\beta)=2-2\cos\alpha\cos\beta+2\text{sen}\alpha\text{sen}\beta\Leftrightarrow\cos(\alpha+\beta)=\cos\alpha\cos\beta-\text{sen}\alpha\text{sen}\beta$$

$$\cos(\alpha-\beta)=\cos(\alpha+(-\beta))=\cos\alpha\cos(-\beta)-\text{sen}\alpha\,\text{sen}(-\beta)=\cos\alpha\cos\beta-\text{sen}\alpha\cdot(-\text{sen}\beta)=$$
$$=\cos\alpha\cos\beta+\text{sen}\alpha\,\text{sen}\beta$$

$$\text{sen}(\alpha+\beta)=\cos\left(\frac{\pi}{2}-(\alpha+\beta)\right)=\cos\left(\left(\frac{\pi}{2}-\alpha\right)-\beta\right)=\cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta+\text{sen}\left(\frac{\pi}{2}-\alpha\right)\text{sen}\beta=$$
$$=\text{sen}\alpha\cos\beta+\text{sen}\beta\cos\alpha$$

$$\text{sen}(\alpha-\beta)=\text{sen}(\alpha+(-\beta))=\text{sen}\alpha\cos(-\beta)+\text{sen}(-\beta)\cos\alpha=\text{sen}\alpha\cos\beta-\text{sen}\beta\cos\alpha$$

As fórmulas a seguir permitem o cálculo da tangente da soma e da diferença de arcos, com $\alpha,\beta,\alpha+\beta,\alpha-\beta\neq\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$.

$$\mathbf{tg(\alpha+\beta)=\frac{tg\alpha+tg\beta}{1-tg\alpha\cdot tg\beta}}$$
$$\mathbf{tg(\alpha-\beta)=\frac{tg\alpha-tg\beta}{1+tg\alpha\cdot tg\beta}}$$

Demonstração:

$$\text{tg}(\alpha+\beta)=\frac{\text{sen}(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\text{sen}\alpha\cos\beta+\text{sen}\beta\cos\alpha}{\cos\alpha\cos\beta-\text{sen}\alpha\,\text{sen}\beta}=\frac{(\text{sen}\alpha\cos\beta+\text{sen}\beta\cos\alpha)\div\cos\alpha\cos\beta}{(\cos\alpha\cos\beta-\text{sen}\alpha\,\text{sen}\beta)\div\cos\alpha\cos\beta}=$$
$$=\frac{\frac{\text{sen}\alpha}{\cos\alpha}+\frac{\text{sen}\beta}{\cos\beta}}{1-\frac{\text{sen}\alpha}{\cos\alpha}\cdot\frac{\text{sen}\beta}{\cos\beta}}=\frac{\text{tg}\alpha+\text{tg}\beta}{1-\text{tg}\alpha\cdot\text{tg}\beta}$$

$$t(\alpha-\beta)=\text{tg}(\alpha+(-\beta))=\frac{\text{tg}\alpha+\text{tg}(-\beta)}{1-\text{tg}\alpha\cdot\text{tg}(-\beta)}=\frac{\text{tg}\alpha-\text{tg}\beta}{1-\text{tg}\alpha\cdot(-\text{tg}\beta)}=\frac{\text{tg}\alpha-\text{tg}\beta}{1+\text{tg}\alpha\cdot\text{tg}\beta}$$

Exemplo: Calcule o seno, o cosseno e a tangente de $15°$.

$$\text{sen}15°=\text{sen}(45°-30°)=\text{sen}45°\cos30°-\text{sen}30°\cos45°=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos15°=\cos(45°-30°)=\cos45°\cos30°+\text{sen}30°\text{sen}45°=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\text{tg}15°=\text{tg}(45°-30°)=\frac{\text{tg}45°-\text{tg}30°}{1+\text{tg}45°\cdot\text{tg}30°}=\frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}}=\frac{3-\sqrt{3}}{3+\sqrt{3}}=\frac{(3-\sqrt{3})^2}{9-3}=\frac{12-6\sqrt{3}}{6}=2-\sqrt{3}$$

OBSERVAÇÃO

A expressão $\mathbf{y = a\,sen\,x + b\cos x}$, onde $\mathbf{a^2 + b^2 \neq 0}$, tem valor mínimo $-\sqrt{\mathbf{a^2 + b^2}}$ e valor máximo $\sqrt{\mathbf{a^2 + b^2}}$.

Demonstração:

$$y = a\,sen\,x + b\cos x = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}sen\,x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$$

Como $\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$ (relação fundamental da trigonometria), então podemos fazer $\frac{a}{\sqrt{a^2+b^2}} = \cos\theta$ e $\frac{b}{\sqrt{a^2+b^2}} = sen\,\theta$. Assim, temos:

$$y = a\,sen\,x + b\cos x = \sqrt{a^2+b^2}(\cos\theta\,sen\,x + sen\,\theta\cos x) = \sqrt{a^2+b^2}\,sen(x+\theta)$$

$$\Rightarrow -\sqrt{a^2+b^2} \le y \le \sqrt{a^2+b^2}$$

2. FÓRMULAS DE ARCO DOBRO E TRIPLO

As fórmulas a seguir permitem calcular o seno, o cosseno e a tangente do dobro de um arco.

$$\mathbf{sen\,2\alpha = 2\,sen\,\alpha \cdot \cos\alpha}$$

$$\mathbf{\cos 2\alpha = \cos^2\alpha - sen^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\,sen^2\alpha}$$

$$\mathbf{tg\,2\alpha = \frac{2\,tg\,\alpha}{1 - tg^2\alpha}}$$

Demonstração:

$$sen\,2\alpha = sen(\alpha+\alpha) = sen\,\alpha\cos\alpha + sen\,\alpha\cos\alpha = 2\,sen\,\alpha\cos\alpha$$

$$\cos 2\alpha = \cos(\alpha+\alpha) = \cos\alpha\cos\alpha - sen\,\alpha\,sen\,\alpha = \cos^2\alpha - sen^2\alpha$$

$$\cos 2\alpha = \cos^2\alpha - sen^2\alpha = \cos^2\alpha - (1-\cos^2\alpha) = 2\cos^2\alpha - 1$$

$$\cos 2\alpha = \cos^2\alpha - sen^2\alpha = (1 - sen^2\alpha) - sen^2\alpha = 1 - 2\,sen^2\alpha$$

$$tg\,2\alpha = tg(\alpha+\alpha) = \frac{tg\,\alpha + tg\,\alpha}{1 - tg\,\alpha \cdot tg\,\alpha} = \frac{2\,tg\,\alpha}{1 - tg^2\alpha}$$

Note que a fórmula de $\text{tg}\,2\alpha$ só é válida se $2\alpha \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

As fórmulas a seguir permitem calcular o seno, o cosseno e a tangente do triplo de um arco.

$$\mathbf{sen\,3\alpha = 3\,sen\,\alpha - 4\,sen^3\,\alpha}$$

$$\mathbf{\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha}$$

$$\mathbf{tg\,3\alpha = \frac{3\,tg\,\alpha - tg^3\,\alpha}{1 - 3\,tg^2\,\alpha}}$$

Demonstração:

$$\text{sen}\,3\alpha = \text{sen}(2\alpha + \alpha) = \text{sen}\,2\alpha\cos\alpha + \text{sen}\,\alpha\cos 2\alpha = 2\,\text{sen}\,\alpha\cos\alpha\cdot\cos\alpha + \text{sen}\,\alpha\cdot\left(1 - 2\,\text{sen}^2\alpha\right) =$$
$$= 2\,\text{sen}\,\alpha\left(1 - \text{sen}^2\alpha\right) + \text{sen}\,\alpha - 2\,\text{sen}^3\alpha = 3\,\text{sen}\,\alpha - 4\,\text{sen}^3\alpha$$

$$\cos 3\alpha = \cos(2\alpha + \alpha) = \cos 2\alpha\cos\alpha - \text{sen}\,2\alpha\,\text{sen}\,\alpha = \left(2\cos^2\alpha - 1\right)\cdot\cos\alpha - 2\,\text{sen}\,\alpha\cos\alpha\cdot\text{sen}\,\alpha =$$
$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha\left(1 - \cos^2\alpha\right) = 4\cos^3\alpha - 3\cos\alpha$$

$$\text{tg}\,3\alpha = \text{tg}(2\alpha + \alpha) = \frac{\text{tg}\,2\alpha + \text{tg}\,\alpha}{1 - \text{tg}\,2\alpha\cdot\text{tg}\,\alpha} = \frac{\frac{2\,\text{tg}\,\alpha}{1 - \text{tg}^2\alpha} + \text{tg}\,\alpha}{1 - \frac{2\,\text{tg}\,\alpha}{1 - \text{tg}^2\alpha}\cdot\text{tg}\,\alpha} = \frac{2\,\text{tg}\,\alpha + \text{tg}\,\alpha - \text{tg}^3\alpha}{1 - \text{tg}^2\alpha - 2\,\text{tg}^2\alpha} = \frac{3\,\text{tg}\,\alpha - \text{tg}^3\alpha}{1 - 3\,\text{tg}^2\alpha}$$

3. FÓRMULAS DE ARCO METADE

As seguintes fórmulas permitem calcular o seno, o cosseno e a tangente da metade de um arco, a menos do sinal.

$$\mathbf{sen\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}}$$

$$\mathbf{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}}$$

$$\mathbf{tg\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}}$$

Demonstração:

$$\cos\alpha = \cos\left(2\cdot\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2} - 1 \Leftrightarrow \cos^2\frac{\alpha}{2} = \frac{\cos\alpha + 1}{2} \Leftrightarrow \cos\frac{\alpha}{2} = \pm\sqrt{\frac{\cos\alpha + 1}{2}}$$

$$\cos\alpha=\cos\left(2\cdot\frac{\alpha}{2}\right)=1-2\,\text{sen}^2\frac{\alpha}{2}\Leftrightarrow\text{sen}^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}\Leftrightarrow\text{sen}\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\text{tg}\frac{\alpha}{2}=\frac{\text{sen}\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}=\frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}}=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Exemplo: Calcule o seno, o cosseno e a tangente de $\frac{\pi}{8}$.

Como $\frac{\pi}{8}\in Q_I$, então todas as suas linhas trigonométricas são positivas.

$$\text{sen}\frac{\pi}{8}=\text{sen}\frac{\pi/4}{2}=\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}=\frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cos\frac{\pi}{8}=\cos\frac{\pi/4}{2}=\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\text{tg}\frac{\pi}{8}=\text{tg}\frac{\pi/4}{2}=\sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}}=\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\cdot\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}=\frac{2-\sqrt{2}}{\sqrt{4-2}}=\sqrt{2}-1$$

4. FÓRMULAS DE DUPLICAÇÃO USANDO TANGENTE

As seguintes fórmulas permitem calcular o seno e o cosseno de um arco conhecendo-se a tangente do seu arco metade.

$$\text{sen}\,x=\frac{2\,\text{tg}\frac{x}{2}}{1+\text{tg}^2\frac{x}{2}}\qquad\qquad \cos x=\frac{1-\text{tg}^2\frac{x}{2}}{1+\text{tg}^2\frac{x}{2}}$$

Demonstração:

$$\text{sen}x=2\,\text{sen}\frac{x}{2}\cos\frac{x}{2}=2\cdot\frac{\text{sen}\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2}=\frac{2\,\text{tg}\frac{x}{2}}{\sec^2\frac{x}{2}}=\frac{2\,\text{tg}\frac{x}{2}}{1+\text{tg}^2\frac{x}{2}}$$

$$\cos x=\cos^2\frac{x}{2}-\operatorname{sen}^2\frac{x}{2}=\left(1-\frac{\operatorname{sen}^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)\cdot\cos^2\frac{x}{2}=\frac{1-\operatorname{tg}^2\frac{x}{2}}{\sec^2\frac{x}{2}}=\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}$$

Exemplo: (UNIFESO 2006) Se x é a medida de um arco do primeiro quadrante e se $\operatorname{sen}x=3\cos x$, então $\operatorname{sen}(2x)$ é igual a

a) $\frac{\sqrt{5}}{5}$ b) $\frac{3}{5}$ c) $\frac{1+\sqrt{5}}{5}$ d) $\frac{4}{5}$ e) $\frac{\sqrt{3}}{2}$

RESOLUÇÃO: b

$$\operatorname{sen}x=3\cos x\Leftrightarrow\operatorname{tg}x=3\Rightarrow\operatorname{sen}2x=\frac{2\operatorname{tg}x}{1+\operatorname{tg}^2x}=\frac{2\cdot3}{1+3^2}=\frac{3}{5}$$

A fórmula seguinte permite calcular a tangente da metade de um ângulo conhecendo-se o seno e o cosseno do ângulo.

$$\mathbf{tg\frac{x}{2}=\frac{sen\,x}{1+\cos x}}$$

Demonstração:

$$\frac{\operatorname{sen}x}{1+\cos x}=\frac{2\operatorname{sen}\frac{x}{2}\cos\frac{x}{2}}{1+\left(2\cos^2\frac{x}{2}-1\right)}=\frac{2\operatorname{sen}\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}=\frac{\operatorname{sen}\frac{x}{2}}{\cos\frac{x}{2}}=\operatorname{tg}\frac{x}{2}$$

Essa relação pode ser facilmente identificada no ciclo trigonométrico para ângulos agudos. Basta fazer $A\hat{O}P=x$, o que implica $A\hat{A}'P=\frac{x}{2}$ (ângulo inscrito) e podemos calcular a $\operatorname{tg}\frac{x}{2}$ dividindo o cateto oposto $\operatorname{sen}x$ pelo cateto adjacente $1+\cos x$.

5. FÓRMULAS DE PROSTAFÉRESE OU DE WERNER

As fórmulas de Prostaférese ou de Werner permitem transformar somas ou diferenças de senos, cossenos e tangentes em produtos ou vice-versa.

As fórmulas a seguir permitem transformar somas e diferenças em produtos.

$$\text{sen}\,p+\text{sen}\,q=2\,\text{sen}\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\text{sen}\,p-\text{sen}\,q=2\,\text{sen}\frac{p-q}{2}\cos\frac{p+q}{2}$$

$$\cos p+\cos q=2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$$

$$\cos p-\cos q=-2\,\text{sen}\frac{p+q}{2}\,\text{sen}\frac{p-q}{2}$$

$$\text{tg}\,p+\text{tg}\,q=\frac{\text{sen}(p+q)}{\cos p\cdot\cos q}$$

$$\text{tg}\,p-\text{tg}\,q=\frac{\text{sen}(p-q)}{\cos p\cdot\cos q}$$

Demonstração:

$$\text{sen}\,p=\text{sen}\left(\frac{p+q}{2}+\frac{p-q}{2}\right)=\text{sen}\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)+\text{sen}\left(\frac{p-q}{2}\right)\cos\left(\frac{p+q}{2}\right)$$

$$\text{sen}\,q=\text{sen}\left(\frac{p+q}{2}-\frac{p-q}{2}\right)=\text{sen}\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)-\text{sen}\left(\frac{p-q}{2}\right)\cos\left(\frac{p+q}{2}\right)$$

$$\Rightarrow \text{sen}\,p+\text{sen}\,q=2\,\text{sen}\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)\ \wedge\ \text{sen}\,p-\text{sen}\,q=2\,\text{sen}\left(\frac{p-q}{2}\right)\cos\left(\frac{p+q}{2}\right)$$

$$\cos p=\cos\left(\frac{p+q}{2}+\frac{p-q}{2}\right)=\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)-\text{sen}\left(\frac{p+q}{2}\right)\text{sen}\left(\frac{p-q}{2}\right)$$

$$\cos q=\cos\left(\frac{p+q}{2}-\frac{p-q}{2}\right)=\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)+\text{sen}\left(\frac{p+q}{2}\right)\text{sen}\left(\frac{p-q}{2}\right)$$

$$\Rightarrow \cos p+\cos q=2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)\ \wedge\ \cos p-\cos q=-2\,\text{sen}\left(\frac{p+q}{2}\right)\text{sen}\left(\frac{p-q}{2}\right)$$

$$\text{tg}\,p+\text{tg}\,q=\frac{\text{sen}\,p}{\cos p}+\frac{\text{sen}\,q}{\cos q}=\frac{\text{sen}\,p\cos q+\text{sen}\,q\cos p}{\cos p\cos q}=\frac{\text{sen}(p+q)}{\cos p\cos q}$$

$$\text{tg}\,p-\text{tg}\,q=\frac{\text{sen}\,p}{\cos p}-\frac{\text{sen}\,q}{\cos q}=\frac{\text{sen}\,p\cos q-\text{sen}\,q\cos p}{\cos p\cos q}=\frac{\text{sen}(p-q)}{\cos p\cos q}$$

As fórmulas a seguir permitem transformar produtos em soma.

$$\mathbf{senp \cdot senq = \frac{1}{2}\left[\cos(p-q) - \cos(p+q)\right]}$$

$$\mathbf{cosp \cdot cosq = \frac{1}{2}\left[\cos(p+q) + \cos(p-q)\right]}$$

$$\mathbf{senp \cdot cosq = \frac{1}{2}\left[sen(p+q) + sen(p-q)\right]}$$

Demonstração:

$$\cos(p-q) - \cos(p+q) = -2sen\frac{(p-q)+(p+q)}{2}sen\frac{(p-q)-(p+q)}{2} = -2senp\,sen(-q) = 2senp\,senq$$

$$\Leftrightarrow senp \cdot senq = \frac{1}{2}\left[\cos(p-q) - \cos(p+q)\right]$$

$$\cos(p+q) + \cos(p-q) = 2\cos\frac{(p+q)+(p-q)}{2}\cos\frac{(p+q)-(p-q)}{2} = 2\cos p\cos q$$

$$\Leftrightarrow \cos p \cdot \cos q = \frac{1}{2}\left[\cos(p+q) + \cos(p-q)\right]$$

$$sen(p+q) + sen(p-q) = 2sen\frac{(p+q)+(p-q)}{2}\cos\frac{(p+q)-(p-q)}{2} = 2senp\cos q$$

$$\Leftrightarrow senp \cdot \cos q = \frac{1}{2}\left[sen(p+q) + sen(p-q)\right]$$

Exemplo: (IME 2012) O valor de $y = sen70^\circ \cos 50^\circ + sen260^\circ \cos 280^\circ$ é:

(A) $\sqrt{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{3}}{3}$ (D) $\frac{\sqrt{3}}{4}$ (E) $\frac{\sqrt{3}}{5}$

RESOLUÇÃO: D

Como $sena \cdot \cos b = \frac{1}{2}\left[sen(a+b) + sen(a-b)\right]$, temos:

$$y = sen70^\circ \cos 50^\circ + sen260^\circ \cos 280^\circ =$$
$$= \frac{1}{2}\left[sen(70^\circ + 50^\circ) + sen(70^\circ - 50^\circ)\right] + \frac{1}{2}\left[sen(260^\circ + 280^\circ) + sen(260^\circ - 280^\circ)\right] =$$

$$=\frac{1}{2}\left(\text{sen}120^\circ+\text{sen}20^\circ\right)+\frac{1}{2}\left(\text{sen}540^\circ+\text{sen}\left(-20^\circ\right)\right)=$$

$$=\frac{1}{2}\left(\text{sen}60^\circ+\text{sen}20^\circ\right)+\frac{1}{2}\left(\text{sen}\left(3\cdot180^\circ\right)-\text{sen}20^\circ\right)=$$

$$=\frac{1}{2}\left(\frac{\sqrt{3}}{2}+\text{sen}20^\circ+0-\text{sen}20^\circ\right)=\frac{\sqrt{3}}{4}$$

REFERÊNCIA: Gelca, R e Andreescu, T. – Putnam and Beyond – pg. 233.

EXERCÍCIOS DE COMBATE

1. Se $\text{sen}(x+y)=m$ e $\text{sen}(x-y)=n$, então $\frac{\text{tg}\,x}{\text{tg}\,y}$ é igual a:

a) $2mn$

b) $\frac{m-n}{2(m+n)}$

c) $\frac{m+n}{m-n}$

d) $\frac{m+n}{2(m-n)}$

e) $\frac{m}{n}$

2. (FUVEST 2001) Se $\text{tg}\,\theta=2$, então o valor de $\frac{\cos 2\theta}{1+\text{sen}2\theta}$ é:

a) -3

b) $\frac{2}{3}$

c) $\frac{1}{3}$

d) $-\frac{1}{3}$

e) $\frac{3}{4}$

3. (EEAr 2003) Se $0<x<\frac{\pi}{2}$, então a expressão $\text{tg}\frac{x}{2}+\text{cotg}\frac{x}{2}$ é equivalente a

a) $2\,\text{sen}\,x$.

b) $2\sec x$.

c) $2\cos x$.

d) $2\,\text{cossec}\,x$.

4. (AFA 2011) O período da função real **f** definida por $f(x)=\dfrac{\text{sen}3x+\text{sen}x}{\cos 3x+\cos x}$ é igual a

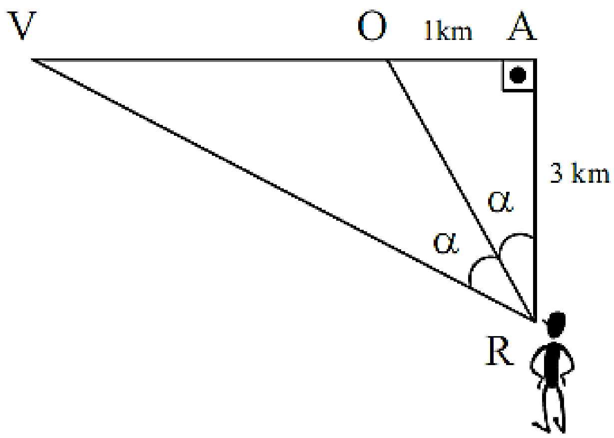
a) 2π

b) π

c) $\dfrac{\pi}{4}$

d) $\dfrac{\pi}{2}$

5. (AFA 2001) Ao saltar do avião que sobrevoa o ponto A (veja figura), um paraquedista cai e toca o solo no ponto V. Um observador que está em R contacta a equipe de resgate localizada em O. A distância, em km, entre o ponto em que o paraquedista tocou o solo e a equipe de resgate é igual a



a) 1,15

b) 1,25

c) 1,35

d) 1,75

6. (AFA 2000) Se $a+b=\dfrac{5\pi}{4}$, então $(1+\text{tg}\,a)(1+\text{tg}\,b)$ é

a) 0

b) 1

c) 2

d) 3

7. (AFA 1999) O valor da expressão $\cos 15° + \text{sen} 105°$ é:

a) $\frac{\sqrt{6}+\sqrt{2}}{4}$

b) $\frac{\sqrt{6}-\sqrt{2}}{4}$

c) $\frac{\sqrt{6}+\sqrt{2}}{2}$

d) $\frac{\sqrt{6}-\sqrt{2}}{2}$

8. (EFOMM 2013) Se $\det\begin{vmatrix} \cos x & \text{sen} x \\ \text{sen} y & \cos y \end{vmatrix} = -\frac{1}{3}$, então o valor de $3\,\text{sen}(x+y)+\text{tg}(x+y)-\sec(x+y)$, para $\frac{\pi}{2} \le x+y \le \pi$, é igual a:

a) 0

b) $\frac{1}{3}$

c) 2

d) 3

e) $\frac{1}{2}$

9. (EFOMM 1999) Sabendo que $\frac{\pi}{2} < \theta < \pi$ e que $\text{sen}\theta = \frac{3}{5}$, o valor de $\cos\left(\frac{\pi}{2}+\theta\right) - \text{sen}(\pi - 2\theta)$ é igual a:

a) $\frac{9}{25}$

b) $-\frac{39}{25}$

c) $2-\sqrt{2}$

d) $\frac{4+\sqrt{5}}{25}$

e) $\frac{3-\sqrt{2}}{9}$

10. (EFOMM 1997) Sabendo-se que $\theta = 67°30'$, o valor de $\text{sen}^4\left(\frac{\theta}{3}\right) + \cos^4\left(\frac{\theta}{3}\right)$ é:

a) $5\sqrt{2}$

b) $\frac{3}{4}$

c) $2\sqrt{\frac{2}{3}}$

d) $\frac{4}{3}$

e) $3\sqrt{\frac{2}{4}}$

11. (EN 2015) O valor do produto $\cos 40° \cdot \cos 80° \cdot \cos 160°$ é

a) $-\frac{1}{8}$

b) $-\frac{1}{4}$

c) -1

d) $-\frac{\sqrt{3}}{2}$

e) $-\frac{\sqrt{2}}{2}$

12. (EN 2015) Um observador, de altura desprezível, situado a 25 m de um prédio, observa-o sob um certo ângulo de elevação. Afastando-se mais 50 m em linha reta, nota que o ângulo de visualização passa a ser a metade do anterior. Podemos afirmar que a altura, em metros, do prédio é

a) $15\sqrt{2}$

b) $15\sqrt{3}$

c) $15\sqrt{5}$

d) $25\sqrt{3}$

e) $25\sqrt{5}$

13. (EN 1998) Sendo $y=\text{sen}\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$, o valor numérico de y é

a) $\frac{1}{2}+\frac{\sqrt{3}}{4}$

b) $\frac{\sqrt{3}}{2}$

c) $\frac{1}{2}$

d) $\sqrt{3}+2$

e) $2\left(\sqrt{3}+1\right)$

14. (EN 1996) Sabendo-se que $\text{tg}x=a$ e $\text{tg}y=b$; pode-se reescrever $Z=\frac{\text{sen}2x+\text{sen}2y}{\text{sen}2x-\text{sen}2y}$ como

a) $\left(\frac{1-ab}{1+ab}\right)\cdot\left(\frac{a-b}{a+b}\right)$

b) $\left(\frac{1+ab}{1-ab}\right)\cdot\left(\frac{a-b}{a+b}\right)$

c) $\left(\frac{1-ab}{1+ab}\right)\cdot\left(\frac{a+b}{a-b}\right)$

d) $\left(\frac{1+ab}{1-ab}\right)\cdot\left(\frac{-a+b}{a-b}\right)$

e) $\left(\frac{1+ab}{1-ab}\right)\cdot\left(\frac{a+b}{a-b}\right)$

15. (EN 1994) Se $\frac{\text{sen}x-\text{sen}y}{\cos x-\cos y}=2$ e $\text{tg}x=\frac{1}{3}$, então $\text{tg}y$ é igual a:

a) 3

b) $\frac{1}{6}$

c) 0

d) $-\frac{1}{6}$

e) -3

16. O valor da expressão $\dfrac{\text{sen}32^\circ+\text{sen}38^\circ+\text{sen}70^\circ}{\cos16^\circ.\cos19^\circ.\cos55^\circ}$ é igual a:

a) 0

b) 1

c) 2

d) 3

e) 4

17. (ITA 1974) A expressão $\left(\dfrac{1-\text{tg}\,x}{1+\text{tg}\,x}\right)^2$ é equivalente a:

a) $\dfrac{1-2\text{sen}2x}{1+\text{sen}2x}$

b) $\dfrac{1+2\text{sen}2x}{1-\text{sen}2x}$

c) $\dfrac{1+\text{sen}2x}{1-\text{sen}2x}$

d) $\dfrac{1-\text{sen}2x}{1+\text{sen}2x}$

e) $\text{tg}\,2x$

18. (ITA 1989) Se $\text{tg}(2A)=5$ então $\text{tg}\left(\dfrac{\pi}{4}+A\right)-\text{tg}\left(\dfrac{\pi}{4}-A\right)$ é igual a:

a) $-\dfrac{40}{21}$

b) -2

c) 5

d) 8

e) 10

19. (ITA 1994) A expressão trigonométrica $\dfrac{1}{\left(\cos^2 x-\text{sen}^2 x\right)^2}-\dfrac{4\,\text{tg}^2 x}{\left(1-\text{tg}^2 x\right)^2}$ para $x\in\left]0,\pi/2\right[$, $x\neq\dfrac{\pi}{4}$, é igual a

a) $\text{sen}(2x)$

b) $\cos(2x)$

c) 1

d) 0

e) $\sec(x)$

20. (ITA 1995) A expressão $\dfrac{\text{sen}\theta}{1+\cos\theta}$, $0<\theta<\pi$, é idêntica a

a) $\sec\dfrac{\theta}{2}$

b) $\text{cossec}\dfrac{\theta}{2}$

c) $\text{cotg}\dfrac{\theta}{2}$

d) $\text{tg}\dfrac{\theta}{2}$

e) $\cos\dfrac{\theta}{2}$

21. (ITA 1996) Seja $\alpha\in\left[0,\dfrac{\pi}{2}\right]$, tal que $\text{sen}\alpha+\cos\alpha=m$. Então, o valor de $y=\dfrac{\text{sen}2\alpha}{\text{sen}^3\alpha+\cos^3\alpha}$ será:

a) $\dfrac{2(m^2-1)}{m(4-m^2)}$

b) $\dfrac{2(m^2+1)}{m(4+m^2)}$

c) $\dfrac{2(m^2-1)}{m(3-m^2)}$

d) $\dfrac{2(m^2-1)}{m(3+m^2)}$

e) $\dfrac{2(m^2+1)}{m(3-m^2)}$

22. (ITA 1999) Se $x\in\left[0,\dfrac{\pi}{2}\right[$ é tal que $4\,\text{tg}^4 x=\dfrac{1}{\cos^4 x}+4$, então o valor de $\text{sen}2x+\text{sen}4x$

a) $\frac{\sqrt{15}}{4}$

b) $\frac{\sqrt{15}}{8}$

c) $\frac{3\sqrt{5}}{8}$

d) $\frac{1}{2}$

e) 1

23. (ITA 1999) Seja $a \in \mathbb{R}$ com $0 < a < \frac{\pi}{2}$. A expressão $\left[\operatorname{sen}\left(\frac{3\pi}{4}+a\right)+\operatorname{sen}\left(\frac{3\pi}{4}-a\right)\right]\operatorname{sen}\left(\frac{\pi}{2}-a\right)$ é idêntica a:

a) $\frac{\sqrt{2}\operatorname{cotg}^2 a}{1+\operatorname{cotg}^2 a}$

b) $\frac{\sqrt{2}\operatorname{cotg} a}{1+\operatorname{cotg}^2 a}$

c) $\frac{\sqrt{2}}{1+\operatorname{cotg}^2 a}$

d) $\frac{1+3\operatorname{cotg} a}{2}$

e) $\frac{1+2\operatorname{cotg} a}{1+\operatorname{cotg} a}$

24. (ITA 2012) Seja $x \in [0, 2\pi]$ tal que $\operatorname{sen}(x)\cos(x) = \frac{2}{5}$. Então, o produto e a soma de todos os possíveis valores de $\operatorname{tg}(x)$ são, respectivamente

a) 1 e 0.

b) 1 e $\frac{5}{2}$.

c) -1 e 0.

d) 1 e 5.

e) -1 e $-\frac{5}{2}$.

GABARITO

1.

$$\begin{cases}\operatorname{sen}(x+y)=m\\ \operatorname{sen}(x-y)=n\end{cases} \Leftrightarrow \begin{cases}\operatorname{sen}x\cos y+\operatorname{sen}y\cos x=m\\ \operatorname{sen}x\cos y-\operatorname{sen}y\cos x=n\end{cases} \Rightarrow \operatorname{sen}x\cos y=\frac{m+n}{2} \wedge \operatorname{sen}y\cos x=\frac{m-n}{2}$$

$$\frac{\operatorname{tg}x}{\operatorname{tg}y}=\frac{\frac{\operatorname{sen}x}{\cos x}}{\frac{\operatorname{sen}y}{\cos y}}=\frac{\operatorname{sen}x\cos y}{\operatorname{sen}y\cos x}=\frac{\frac{m+n}{2}}{\frac{m-n}{2}}=\frac{m+n}{m-n}$$

RESPOSTA: C

2.

$$\frac{\cos 2\theta}{1+\operatorname{sen}2\theta}=\frac{\frac{1-\operatorname{tg}^2\theta}{1+\operatorname{tg}^2\theta}}{1+\frac{2\operatorname{tg}\theta}{1+\operatorname{tg}^2\theta}}=\frac{1-\operatorname{tg}^2\theta}{1+\operatorname{tg}^2\theta+2\operatorname{tg}\theta}=\frac{(1+\operatorname{tg}\theta)(1-\operatorname{tg}\theta)}{(1+\operatorname{tg}\theta)^2}=\frac{1-\operatorname{tg}\theta}{1+\operatorname{tg}\theta}=\frac{1-2}{1+2}=-\frac{1}{3}$$

RESPOSTA: D

3.

$$\operatorname{tg}\frac{x}{2}+\operatorname{cotg}\frac{x}{2}=\frac{\operatorname{sen}\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\cos\frac{x}{2}}{\operatorname{sen}\frac{x}{2}}=\frac{\operatorname{sen}^2\frac{x}{2}+\cos^2\frac{x}{2}}{\operatorname{sen}\frac{x}{2}\cos\frac{x}{2}}=\frac{1}{\operatorname{sen}\frac{x}{2}\cos\frac{x}{2}}=$$

$$=\frac{2}{2\operatorname{sen}\frac{x}{2}\cos\frac{x}{2}}=\frac{2}{\operatorname{sen}x}=2\operatorname{cossec}x$$

RESPOSTA: D

4.

$$f(x)=\frac{\operatorname{sen}3x+\operatorname{sen}x}{\cos 3x+\cos x}=\frac{2\operatorname{sen}2x\cos x}{2\cos 2x\cos x}=\operatorname{tg}2x$$

Como o período da função $\operatorname{tg}x$ é π, então o período da função $f(x)=\operatorname{tg}2x$ é $\frac{\pi}{2}$.

RESPOSTA: D

5.

$$\operatorname{tg}\alpha=\frac{1}{3}\Rightarrow \operatorname{tg}2\alpha=\frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}=\frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}=\frac{\frac{2}{3}}{\frac{8}{9}}=\frac{3}{4}$$

$$\operatorname{tg}2\alpha=\frac{VO+1}{3}=\frac{3}{4}\Leftrightarrow VO=\frac{5}{4}=1{,}25\,\text{km}$$

RESPOSTA: B

6.

$$\operatorname{tg}(a+b)=\operatorname{tg}\frac{5\pi}{4}=1\Rightarrow\frac{\operatorname{tg}a+\operatorname{tg}b}{1-\operatorname{tg}a\operatorname{tg}b}=1\Leftrightarrow \operatorname{tg}a+\operatorname{tg}b=1-\operatorname{tg}a\operatorname{tg}b$$

$$\Leftrightarrow \operatorname{tg}a+\operatorname{tg}b+\operatorname{tg}a\operatorname{tg}b+1=2\Leftrightarrow(1+\operatorname{tg}a)(1+\operatorname{tg}b)=2$$

RESPOSTA: C

7.

$$\cos15^\circ+\operatorname{sen}105^\circ=\cos15^\circ+\operatorname{sen}(90^\circ+15^\circ)=\cos15^\circ+\cos15^\circ=2\cos15^\circ=2\cdot\frac{\sqrt{6}+\sqrt{2}}{4}=\frac{\sqrt{6}+\sqrt{2}}{2}$$

RESPOSTA: C

8.

$$\det\begin{vmatrix}\cos x & \operatorname{sen}x\\ \operatorname{sen}y & \cos y\end{vmatrix}=-\frac{1}{3}\Leftrightarrow \cos x\cos y-\operatorname{sen}x\operatorname{sen}y=-\frac{1}{3}\Leftrightarrow\cos(x+y)=-\frac{1}{3}$$

$$\frac{\pi}{2}\le x+y\le\pi\Rightarrow\operatorname{sen}(x+y)=\sqrt{1-\cos^2(x+y)}=\sqrt{1-\left(-\frac{1}{3}\right)^2}=\frac{2\sqrt{2}}{3}$$

$$\operatorname{tg}(x+y)=\frac{\operatorname{sen}(x+y)}{\cos(x+y)}=\frac{\frac{2\sqrt{2}}{3}}{-\frac{1}{3}}=-2\sqrt{2}$$

$$\sec(x+y)=\frac{1}{\cos(x+y)}=\frac{1}{-\frac{1}{3}}=-3$$

$$3\operatorname{sen}(x+y)+\operatorname{tg}(x+y)-\sec(x+y)=3\cdot\frac{2\sqrt{2}}{3}+\left(-2\sqrt{2}\right)-(-3)=3$$

RESPOSTA: D

9.

$$\frac{\pi}{2} < \theta < \pi : \cos\theta = -\sqrt{1-\text{sen}^2\,\theta} = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\frac{4}{5}$$

$$\text{sen}2\theta = 2\text{sen}\theta\cos\theta = 2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos\left(\frac{\pi}{2}+\theta\right) - \text{sen}(\pi-2\theta) = -\text{sen}\theta - \text{sen}2\theta = -\frac{3}{5} - \left(-\frac{24}{25}\right) = \frac{9}{25}$$

RESPOSTA: A

10.

$$\text{sen}^4\left(\frac{\theta}{3}\right) + \cos^4\left(\frac{\theta}{3}\right) = \left[\text{sen}^2\left(\frac{\theta}{3}\right) + \cos^2\left(\frac{\theta}{3}\right)\right]^2 - 2\text{sen}^2\left(\frac{\theta}{3}\right)\cos^2\left(\frac{\theta}{3}\right) =$$

$$= 1^2 - \frac{1}{2}\left[2\text{sen}\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right]^2 = 1 - \frac{1}{2}\left[\text{sen}\left(\frac{2\theta}{3}\right)\right]^2 = 1 - \frac{1}{2}\text{sen}^2\left(\frac{2\cdot 67°30'}{3}\right) =$$

$$= 1 - \frac{1}{2}\text{sen}^2 45° = 1 - \frac{1}{2}\cdot\left(\frac{\sqrt{2}}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

RESPOSTA: B

11.

$$y = \cos 40^\circ \cdot \cos 80^\circ \cdot \cos 160^\circ = \cos 40^\circ \cdot \cos 80^\circ \cdot \left(-\cos 20^\circ\right)$$

$$\Leftrightarrow 2\text{sen}20^\circ y = -2\text{sen}20^\circ \cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = -\text{sen}40^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ$$

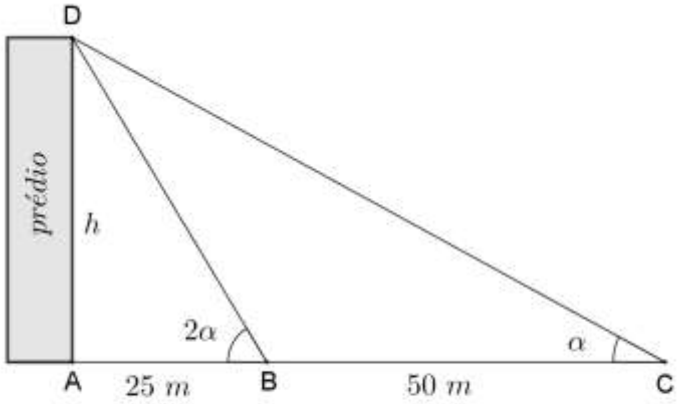
$$\Leftrightarrow 4\text{sen}20^\circ y = -2\text{sen}40^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = -\text{sen}80^\circ \cdot \cos 80^\circ$$

$$\Leftrightarrow 8\text{sen}20^\circ y = -2\text{sen}80^\circ \cdot \cos 80^\circ = -\text{sen}160^\circ = -\text{sen}20^\circ$$

$$\Leftrightarrow y = -\frac{1}{8}$$

RESPOSTA: A

12.



Na figura, temos $\text{tg}\,2\alpha = \frac{h}{25}$ e $\text{tg}\,\alpha = \frac{h}{75}$.

Como $\text{tg}\,2\alpha = \frac{2\,\text{tg}\,\alpha}{1-\text{tg}^2\,\alpha}$, então temos:

$$\frac{h}{25} = \frac{2\cdot\frac{h}{75}}{1-\left(\frac{h}{75}\right)^2} \Leftrightarrow \frac{1}{25} = \frac{\frac{2}{75}}{1-\frac{h^2}{75^2}} \Leftrightarrow \frac{1}{25} = \frac{2}{75}\cdot\frac{75^2}{75^2-h^2} \Leftrightarrow 75^2-h^2 = 150\cdot 25 \Leftrightarrow h^2 = 1875$$

$$\Leftrightarrow h = 25\sqrt{3}\text{ m}$$

RESPOSTA: D

13.

$$y = \text{sen}\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) = \frac{1}{2}\left[\text{sen}\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)+\text{sen}\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)\right] = \frac{1}{2}\left[\text{sen}\frac{\pi}{2}+\text{sen}\frac{\pi}{3}\right] = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) = \frac{1}{2}+\frac{\sqrt{3}}{4}$$

RESPOSTA: A

14.

$$Z = \frac{\text{sen}2x+\text{sen}2y}{\text{sen}2x-\text{sen}2y} = \frac{2\,\text{sen}(x+y)\cos(x-y)}{2\,\text{sen}(x-y)\cos(x+y)} = \frac{\text{tg}(x+y)}{\text{tg}(x-y)} = \frac{\frac{\text{tg}\,x+\text{tg}\,y}{1-\text{tg}\,x\,\text{tg}\,y}}{\frac{\text{tg}\,x-\text{tg}\,y}{1+\text{tg}\,x\,\text{tg}\,y}} =$$

$$= \left(\frac{1+\text{tg}\,x\,\text{tg}\,y}{1-\text{tg}\,x\,\text{tg}\,y}\right)\left(\frac{\text{tg}\,x+\text{tg}\,y}{\text{tg}\,x-\text{tg}\,y}\right) = \left(\frac{1+ab}{1-ab}\right)\cdot\left(\frac{a+b}{a-b}\right)$$

RESPOSTA: E

15.

$$\frac{\operatorname{sen}x-\operatorname{sen}y}{\cos x-\cos y}=2\Leftrightarrow\frac{2\operatorname{sen}\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)}{-2\operatorname{sen}\left(\frac{x+y}{2}\right)\operatorname{sen}\left(\frac{x-y}{2}\right)}=2\Leftrightarrow\operatorname{tg}\left(\frac{x+y}{2}\right)=-\frac{1}{2}$$

$$\Rightarrow\operatorname{tg}(x+y)=\frac{2\operatorname{tg}\left(\frac{x+y}{2}\right)}{1-\operatorname{tg}^2\left(\frac{x+y}{2}\right)}=\frac{2\cdot\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)^2}=-\frac{4}{3}$$

$$\Rightarrow\operatorname{tg}(x+y)=\frac{\operatorname{tg}x+\operatorname{tg}y}{1-\operatorname{tg}x\operatorname{tg}y}\Leftrightarrow-\frac{4}{3}=\frac{\frac{1}{3}+\operatorname{tg}y}{1-\frac{1}{3}\cdot\operatorname{tg}y}\Leftrightarrow\operatorname{tg}y=-3$$

RESPOSTA: E

16.

$$\frac{\operatorname{sen}32^\circ+\operatorname{sen}38^\circ+\operatorname{sen}70^\circ}{\cos16^\circ\cdot\cos19^\circ\cdot\cos55^\circ}=\frac{2\operatorname{sen}\left(\frac{38^\circ+32^\circ}{2}\right)\cos\left(\frac{38^\circ-32^\circ}{2}\right)+2\operatorname{sen}35^\circ\cos35^\circ}{\cos16^\circ\cdot\cos19^\circ\cdot\operatorname{sen}35^\circ}=$$

$$=\frac{2\operatorname{sen}35^\circ\cos3^\circ+2\operatorname{sen}35^\circ\cos35^\circ}{\cos16^\circ\cdot\cos19^\circ\cdot\operatorname{sen}35^\circ}=\frac{2\operatorname{sen}35^\circ\left(\cos3^\circ+\cos35^\circ\right)}{\cos16^\circ\cdot\cos19^\circ\cdot\operatorname{sen}35^\circ}=$$

$$=\frac{2\cdot2\cos\left(\frac{35^\circ+3^\circ}{2}\right)\cos\left(\frac{35^\circ-3^\circ}{2}\right)}{\cos16^\circ\cdot\cos19^\circ}=\frac{4\cos19^\circ\cos16^\circ}{\cos16^\circ\cdot\cos19^\circ}=4$$

RESPOSTA: E

17.

$$\left(\frac{1-\operatorname{tg}x}{1+\operatorname{tg}x}\right)^2=\left(\frac{1-\frac{\operatorname{sen}x}{\cos x}}{1+\frac{\operatorname{sen}x}{\cos x}}\right)^2=\left(\frac{\cos x-\operatorname{sen}x}{\cos x+\operatorname{sen}x}\right)^2=\frac{\cos^2x+\operatorname{sen}^2x-2\operatorname{sen}x\cos x}{\cos^2x+\operatorname{sen}^2x+2\operatorname{sen}x\cos x}=\frac{1-\operatorname{sen}2x}{1+\operatorname{sen}2x}$$

RESPOSTA: D

18.

$$\operatorname{tg}\left(\frac{\pi}{4}+A\right)-\operatorname{tg}\left(\frac{\pi}{4}-A\right)=\frac{\operatorname{tg}\frac{\pi}{4}+\operatorname{tg}A}{1-\operatorname{tg}\frac{\pi}{4}\operatorname{tg}A}-\frac{\operatorname{tg}\frac{\pi}{4}-\operatorname{tg}A}{1+\operatorname{tg}\frac{\pi}{4}\operatorname{tg}A}=\frac{1+\operatorname{tg}A}{1-\operatorname{tg}A}-\frac{1-\operatorname{tg}A}{1+\operatorname{tg}A}=$$

$$=\frac{1+2\operatorname{tg}A+\operatorname{tg}^2A-1+2\operatorname{tg}A-\operatorname{tg}^2A}{1-\operatorname{tg}^2A}=\frac{4\operatorname{tg}A}{1-\operatorname{tg}^2A}=2\cdot\frac{2\operatorname{tg}A}{1-\operatorname{tg}^2A}=2\cdot\operatorname{tg}(2A)=2\cdot5=10$$

RESPOSTA: E

19.

$$\frac{1}{\left(\cos^2x-\operatorname{sen}^2x\right)^2}-\frac{4\operatorname{tg}^2x}{\left(1-\operatorname{tg}^2x\right)^2}=\frac{1}{(\cos2x)^2}-\left(\frac{2\operatorname{tg}x}{1-\operatorname{tg}^2x}\right)^2=\sec^2 2x-\operatorname{tg}^2 2x=\left(1+\operatorname{tg}^2 2x\right)-\operatorname{tg}^2 2x=1$$

RESPOSTA: C

20.

$$\frac{\operatorname{sen}\theta}{1+\cos\theta}=\frac{2\operatorname{sen}\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}=\frac{\operatorname{sen}\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\operatorname{tg}\frac{\theta}{2}$$

RESPOSTA: D

21.

$$y=\frac{\operatorname{sen}2\alpha}{\operatorname{sen}^3\alpha+\cos^3\alpha}=$$

$$=\frac{2\operatorname{sen}\alpha\cos\alpha}{(\operatorname{sen}\alpha+\cos\alpha)\left(\operatorname{sen}^2\alpha-\operatorname{sen}\alpha\cos\alpha+\cos^2\alpha\right)}=$$

$$=\frac{2\operatorname{sen}\alpha\cos\alpha}{(\operatorname{sen}\alpha+\cos\alpha)(1-\operatorname{sen}\alpha\cos\alpha)}$$

$$\operatorname{sen}\alpha+\cos\alpha=m$$

$$\Rightarrow\operatorname{sen}^2\alpha+2\operatorname{sen}\alpha\cos\alpha+\cos^2\alpha=m^2$$

$$\Leftrightarrow\operatorname{sen}\alpha\cos\alpha=\frac{m^2-1}{2}$$

$$\Leftrightarrow y=\frac{m^2-1}{m\cdot\left(1-\frac{m^2-1}{2}\right)}=\frac{2\left(m^2-1\right)}{m\left(3-m^2\right)}$$

RESPOSTA: C

22.

$$4\,\text{tg}^4\,x=\frac{1}{\cos^4 x}+4 \Leftrightarrow 4\cdot\frac{\text{sen}^4\,x}{\cos^4 x}=\frac{1}{\cos^4 x}+4 \Leftrightarrow 4\,\text{sen}^4\,x=1+4\cos^4 x \Leftrightarrow 4\left(\cos^4 x-\text{sen}^4\,x\right)=-1$$

$$\Leftrightarrow 4\left(\cos^2 x+\text{sen}^2\,x\right)\left(\cos^2 x-\text{sen}^2\,x\right)=-1 \Leftrightarrow 4\cos 2x=-1 \Leftrightarrow \cos 2x=-\frac{1}{4}$$

$$x\in\left[0,\frac{\pi}{2}\right[\Rightarrow 2x\in\left[0,\pi\right[\Rightarrow \text{sen}2x>0 \Rightarrow \text{sen}2x=\sqrt{1-\cos^2 2x}=\sqrt{1-\left(-\frac{1}{4}\right)^2}=\frac{\sqrt{15}}{4}$$

$$\Rightarrow \text{sen}4x=2\,\text{sen}2x\cos 2x=2\cdot\frac{\sqrt{15}}{4}\cdot\left(-\frac{1}{4}\right)=-\frac{\sqrt{15}}{8}$$

$$\Rightarrow \text{sen}2x+\text{sen}4x=\frac{\sqrt{15}}{4}+\left(-\frac{\sqrt{15}}{8}\right)=\frac{\sqrt{15}}{8}$$

RESPOSTA: B

23.

$$\left[\text{sen}\left(\frac{3\pi}{4}+a\right)+\text{sen}\left(\frac{3\pi}{4}-a\right)\right]\text{sen}\left(\frac{\pi}{2}-a\right)=2\,\text{sen}\left(\frac{\frac{3\pi}{4}+a+\frac{3\pi}{4}-a}{2}\right)\cos\left(\frac{\frac{3\pi}{4}+a-\left(\frac{3\pi}{4}-a\right)}{2}\right)\cos a=$$

$$=2\,\text{sen}\frac{3\pi}{4}\cos a\cdot\cos a=2\cdot\frac{\sqrt{2}}{2}\cos^2 a=\sqrt{2}\cos^2 a=$$

$$=\sqrt{2}\cdot\frac{1}{\sec^2 a}=\sqrt{2}\cdot\frac{1}{1+\text{tg}^2\,a}=\sqrt{2}\cdot\frac{1}{1+\frac{1}{\text{cotg}^2\,a}}=\frac{\sqrt{2}\,\text{cotg}^2\,a}{1+\text{cotg}^2\,a}$$

RESPOSTA: A

24.

$$\text{sen}(x)\cos(x)=\frac{2}{5} \Leftrightarrow 2\,\text{sen}(x)\cos(x)=\frac{4}{5} \Leftrightarrow \text{sen}(2x)=\frac{4}{5}$$

$$\text{sen}(2x)=\frac{2\,\text{tg}(x)}{1+\text{tg}^2(x)}=\frac{4}{5} \Leftrightarrow 2\,\text{tg}^2(x)-5\,\text{tg}(x)+2=0 \Leftrightarrow \text{tg}(x)=\frac{1}{2} \ \vee\ \text{tg}(x)=2$$

Logo, o produto e a soma de todos os possíveis valores de $\text{tg}(x)$ são, respectivamente, 1 e $\frac{5}{2}$.

RESPOSTA: B